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Hadamard States and Two-dimensional Gravity

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Abstract

We have used a two-dimensional analog of the Hadamard state-condition to study the local constraints on the two-point function of a linear quantum field conformally coupled to a two-dimensional gravitational background. We develop a dynamical model in which the determination of the state of the quantum field is essentially related to the determination of a conformal frame. A particular conformal frame is then introduced in which a two-dimensional gravitational equation is established.

1 Introduction

The subject of quantum field theory in curved spacetime studies the interrelation between quantum theory and spacetime geometry in the approximation that the geometry is remained as a classical background. This subject, however, contains some basic problems concerning inherent ambiguities in the definition of physical states associated with a quantum field. In the absence of a gravitational background, it is always possible to use the Poincare symmetries to obtain a physical vacuum (the state of lowest energy). One may then assume that the physically admissible states most naturally arise as local excitations of this state. In the presence of a curved spacetime, however, this procedure does not apply, because on a general curved sapcetime one may not find a global symmetry. In this case the problem concerning the determination of the local states and the role of the global features of spacetime is of obvious importance.

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The Hadamard state-condition provides a framework in which we may improve our understanding in this context. In this framework one postulates that the short-distance singularity structure of the two-point function of a linear quantum field is represented by the Hadamard ansatz [1]. In this prescription, however, there exist problems in the specification of the state-dependent part of the two-point function. For characterizing the physical states, it therefore seems to be essential to find out a suitable scheme for the treatment of these problems. In the present paper we shall consider this issue for a quantum scalar field conformaly coupled to a two-dimensional gravitational background.
The organization of this paper is as follows: In section 2, we use the Hadamard ansatz for the derivation of the local constraints imposed on the state-dependent part of the two-point function. In section 3, we develop a dynamical model in order to analyze these constraints. This dynamical model, which is a two-dimensional analog of a model considered in the previous publication [2], uses a conformal invariant C-number field to characterize the stress-tensor associated with the local states. The introduction of this scalar field into the analysis leads to a basic connection between the specification of the local states and the determination of a conformal frame. There is a general consistency requirement on the behavior of the C-number field which necessitates the use of a conformal frame in which the trace of the stress-tensor is measured by a cosmological constant. This leads to the establishment of a two-dimensional gravitational equation in this frame. In section 4, we offer some concluding remarks. Throughout the following we shall work in units in which $\hbar = c = 1$ and follow the sign conventions of Hawking and Ellis [3].

2 Hadamard states in two spacetime dimensions

We consider a free massless quantum scalar field $\phi(x)$ propagating in a two-dimensional gravitational background with the action functional (In the following the semicolon and ∇ denote covariant differentiation)

$$S[\phi] = -\frac{1}{2}\int d^2x \sqrt{-g} \;\; g_{\alpha\beta}\nabla^{\alpha}\phi\nabla^{\beta}\phi \; . \tag{1}$$

This action gives rise to the field equation

$$\Box\phi(x) = 0 \; . \tag{2}$$

A state of $\phi(x)$ is characterized by a hierarchy of wightman functions

$$\langle\phi(x_1), ..., \phi(x_n)\rangle \; . \tag{3}$$

We are primarily interested in quasi-free states, i.e., states for which the truncated n-point functions vanish for $n > 2$. Such states may be characterized by their two-point functions. In a linear theory the antisymmetric part of the two-point function is common to all states in the same representation. It is just the universal commutator function. Thus, in our case all the relevant information about the state-dependent part of the two-point function are encoded in its symmetric part, denoted in the following by $G^+(x, x')$, which satisfies Eq.(2) in each argument. Equivalence principle suggests that the leading singularity of $G^+(x, x')$ should

have a close correspondence to the singularity structure of the two-point function of a free massless field in a two-dimensional Minkowski spacetime. One may therefore assume that for x sufficiently close to x' the function $G^+(x,x')$ can be written as

$$G^+(x,x') = -\frac{1}{4\pi}\ln\sigma(x,x') + F(x,x') \ , \tag{4}$$

where $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x', and $F(x,x')$ is a regular function. This may be viewed as a two-dimensional analog of the Hadamard ansatz[1]. It should be remarked that in general there is a missing mass scale in the expression (4) emerging from the fact that the argument of the logarithm must be dimensionless. We shall return to this point in the next section.
The function $F(x,x')$ satisfies a general constraint obtaining from the symmetry condition of $G^+(x,x')$ and requiring that the expression (4) satisfies the wave equation (2). The study of this constraint has obviously a particular significance for analyzing the state-dependent part of the two-point function. We therefore present its derivation as follows: Applying Eq.(2) on (4) leads to

$$\sigma(x,x')\Box F(x,x') = \frac{1}{4\pi}(\sigma^{;\alpha}{}_{;\alpha} - 2) \ , \tag{5}$$

in which we have used the definition of $\sigma(x,x')$, namely

$$\sigma(x,x') = \frac{1}{2}g_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta} \ . \tag{6}$$

The explicit form of the right hand side of (5) can be specified by writing the covariant expansion of $\sigma^{;\alpha\beta}$ [4]

$$\sigma^{;\alpha\beta} = g^{\alpha\beta} - \frac{1}{3}R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}\,\sigma^{;\gamma}\sigma^{;\delta} + \frac{1}{12}R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta;\rho}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\rho} + O(\sigma^2) \ . \tag{7}$$

Taking the trace of this relation and using the fact that in a two-dimensional spacetime the Einstein tensor identically vanishes

$$R_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}R \ , \tag{8}$$

leads to

$$\sigma^{;\alpha}{}_{;\alpha} = 2 - \frac{1}{6}g_{\gamma\delta}R\sigma^{;\gamma}\sigma^{;\delta} + \frac{1}{24}g_{\gamma\delta}R_{;\rho}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\rho} + O(\sigma^2) \ . \tag{9}$$

Combining (5) and (9), yields

$$\sigma(x,x')\Box F(x,x') = -\frac{1}{24\pi}g_{\gamma\delta}R\sigma^{;\gamma}\sigma^{;\delta} + \frac{1}{96\pi}g_{\gamma\delta}R_{;\rho}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\rho} + O(\sigma^2) \ . \tag{10}$$

Substituting (6) and the covariant expansion of $F(x,x')$ [5], namely

$$F(x,x') = F(x) - \frac{1}{2}F_{;\alpha}(x)\sigma^{;\alpha} + \frac{1}{2}F_{\alpha\beta}(x)\sigma^{;\alpha}\sigma^{;\beta} + \frac{1}{4}\{\frac{1}{6}F_{;\alpha\beta\gamma}(x) - F_{\alpha\beta;\gamma}(x)\}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + O(\sigma^2) \ , \tag{11}$$

into (10), one finds

$$\frac{1}{2}F^{\eta}{}_{\eta}(x)g_{\gamma\delta}\sigma^{;\gamma}\sigma^{;\delta}+\frac{1}{2}g_{\gamma\delta}[F_{\alpha\beta}{}^{;\alpha}(x)-\frac{1}{2}F^{\eta}{}_{\eta;\beta}(x)+\frac{1}{12}(\Box F(x))_{;\beta}-\frac{1}{3}\Box(F_{;\beta}(x))-\frac{1}{12}RF_{;\beta}(x)]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\beta}+O(\sigma^2)$$

$$= -\frac{1}{24\pi}g_{\gamma\delta}R\sigma^{;\gamma}\sigma^{;\delta} + \frac{1}{96\pi}g_{\gamma\delta}R_{;\beta}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\beta} + O(\sigma^2) \ . \tag{12}$$

We may now compare term by term up to the third order of $\sigma^{;\alpha}$ to obtain the desired relations

$$F^{\eta}{}_{\eta}(x) = -\frac{1}{12\pi}R \ , \tag{13}$$

$$F_{\alpha\beta}{}^{;\alpha}(x) - \frac{1}{2}F^{\eta}{}_{\eta;\beta}(x) + \frac{1}{12}(\Box F(x))_{;\beta} - \frac{1}{3}\Box(F_{;\beta}(x)) - \frac{1}{12}RF_{;\beta}(x) = \frac{1}{48\pi}R_{;\beta} \ . \tag{14}$$

The equations (13) and (14) are to be regarded as two independent constraints imposed on the state-dependent part of the two-point function. We may combine them to obtain

$$F_{\alpha\beta}{}^{;\alpha}(x) + \frac{1}{12}(\Box F(x))_{;\beta} - \frac{1}{3}\Box(F_{;\beta}(x)) - \frac{1}{12}RF_{;\beta}(x) = -\frac{1}{48\pi}R_{;\beta} \ . \tag{15}$$

It should be noted that in the derivation of this constraint we have used the covariant expansion of $F(x,x')$ and $\sigma^{;\alpha\beta}$ only up to the second order terms. In general there exist additional constraints on the higher order expansion terms. However in our analysis we shall neglect these higher order constraints and concentrate our attention on (15).

3 The large scale behavior of local states

In the first step of analyzing the constraint (15) we note that it can be written as a total divergence. In fact we may use the relation (8) and the differential identity

$$\Box(F_{;\beta}(x)) = (\Box F(x))_{;\beta} + R_{\alpha\beta}F^{;\alpha}(x) \ , \tag{16}$$

to get the equation

$$\nabla^{\alpha}\Sigma_{\alpha\beta} = 0 \ , \tag{17}$$

where

$$\Sigma_{\alpha\beta} = \Sigma^{(0)}_{\alpha\beta} + \Sigma^{(1)}_{\alpha\beta} \ , \tag{18}$$

and

$$\Sigma^{(0)}_{\alpha\beta} = \frac{1}{2}(F_{;\alpha\beta}(x) - \frac{1}{2}g_{\alpha\beta}\Box F(x)) - (F_{\alpha\beta} + \frac{1}{24\pi}g_{\alpha\beta}R) \ , \tag{19}$$

$$\Sigma^{(1)}_{\alpha\beta} = \frac{1}{48\pi}g_{\alpha\beta}R \ . \tag{20}$$

The tensor $\Sigma_{\alpha\beta}$ is decomposed into a traceless part, which is denoted by $\Sigma^{(0)}_{\alpha\beta}$, and the tensor $\Sigma^{(1)}_{\alpha\beta}$ which leads to the trace anomaly in two-dimensions[6, 7]. The principal strategy now is to

relate the tensor $\Sigma^{(0)}_{\alpha\beta}$ to the traceless stress-tensor of a conformal invariant scalar field which is given by [6]

$$T_{\alpha\beta} = \nabla_\alpha\psi\nabla_\beta\psi - \frac{1}{2}g_{\alpha\beta}\nabla_\gamma\psi\nabla^\gamma\psi \ , \tag{21}$$

where $\psi(x)$ is a C-number scalar field satisfying the dynamical equation

$$\Box\psi(x) = 0 \ . \tag{22}$$

In this case the tensor $\Sigma_{\alpha\beta}$ takes the form

$$\Sigma_{\alpha\beta} = T_{\alpha\beta} + \frac{1}{48\pi}g_{\alpha\beta}R \ . \tag{23}$$

This relation indicates how the local states are related to the background geometry. Taking its trace we obtain

$$\Sigma^\alpha_\alpha = \frac{1}{24\pi}R \ , \tag{24}$$

which is the trace anomaly of a two-dimensional quantum stress-tensor of a conformal invariant scalar field. This raises the possibility of regarding $\Sigma_{\alpha\beta}$ as the quantum stress-tensor induced by the two-point function. However by comparing (23) with (17) we observe that the existence of the trace anomaly on the background metric is in conflict with the interpretation of $T_{\alpha\beta}$ as a conserved stress-tensor. This observation reveals that the connection between the local properties of the state-dependent part of the two-point function with those of a C-number scalar field, $\psi(x)$, can not be consistently established on the background metric. This apparent discrepancy, however, may be removed by appealing to the conformal symmetry of the above model. The conformal invariance of Eq.(22) implies that at dynamical level it is not possible to single out a particular configuration for $\psi(x)$ among many different conformally related configurations. In order to determine a configuration one has to choose a particular conformal frame. This therefore raises the question of which of these conformal frames corresponds to the physical one. In general the choice of a conformal frame depends on the physical conditions one wishes to impose on such a frame and these conditions are suggested by the problem under consideration. Here we shall choose the conformal frame by demanding that $T_{\alpha\beta}$ shall be consistent with the conservation property of a stress-tensor. We first consider a conformal transformation

$$g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta} \ , \tag{25}$$

$$\psi(x) \rightarrow \psi(x) \ , \tag{26}$$

where Ω is a nonvanishing, smooth spacetime function. Then we write Eq.(23) in a conformal frame describing by the metric $\bar{g}_{\alpha\beta}$ so that

$$\bar{\Sigma}_{\alpha\beta} = \bar{T}_{\alpha\beta} + \frac{1}{48\pi}\bar{g}_{\alpha\beta}\Lambda \ , \tag{27}$$

where we have set

$$\bar{R} - \Lambda = 0 \ , \tag{28}$$

or, equivalently,

$$\nabla_\gamma \ln\Omega\nabla^\gamma \ln\Omega = \frac{1}{2}\Lambda\Omega^2 - \frac{1}{2}R + \frac{\Box\Omega}{\Omega} \ . \tag{29}$$

This is a differential equation which determines a particular conformal factor characterizing a conformal frame in which the relation (27) holds. Equation (28) corresponds to a two-dimensional analog of the vacuum Einstein field equations with a cosmological constant[8]. This clearly emphasizes the significance of the gravitational coupling of local states in the choice of a conformal frame as a physical one. We should also remark that Λ provides now the missing mass-scale mentioned previously in connection with the Hadamard ansatz. This mass-scale presents itself through the root of an effective cosmological constant in a two-dimensional gravitational theory.
To make a closer look at the relation (29), let us investigate it for a particular case in which the background geometry is described by a two-dimensional Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2. \tag{30}$$

In this case the trace anomaly is $\Sigma^\alpha_\alpha = \frac{GM}{6\pi r^3}$. To solve Eq.(29) in this particular case one needs some boundary conditions. One may intuitively expect that for a sufficiently small cosmological constant the conformal frame, characterized by (29), and the asymptotic regions of the background metric should have the same properties . We therefore choose the boundary conditions so that an asymptotic correspondence can be established between the conformal frame and the background frame in the Schwarzschild coordinate system. Since the trace anomaly vanishes when $r \to \infty$, Eq.(29) reduces to

$$\nabla_\gamma \ln \Omega \nabla^\gamma \ln \Omega = \frac{1}{2}\Lambda\Omega^2 + \frac{\Box\Omega}{\Omega} \ . \tag{31}$$

For a sufficiently small Λ and in a static case, in which Ω is only a function of r, this equation has a simple solution, namely $\Omega(r) = e^{(ar+b)}$ with a and b being arbitrary constants. Applying this conformal factor to (30) gives in the asymptotic regions

$$ds^2 = e^{2(ar+b)}(-dt^2 + dr^2) \ . \tag{32}$$

The case $a = b = 0$ corresponds to the Minkowski spacetime and the conformal frame meets the background metric for $r \to \infty$ in the Schwarzschild coordinate system. Thus $a = b = 0$ characterizes the physically admissible conformal factor. When $a \neq 0$, a coordinate transformation

$$\begin{aligned} \eta &= \frac{1}{a}e^{(ar+b)}\sinh(at) \ , \\ \xi &= \frac{1}{a}e^{(ar+b)}\cosh(at) \ , \end{aligned} \tag{33}$$

would change (32) to

$$ds^2 = -d\eta^2 + d\xi^2 \ . \tag{34}$$

demonstrating that the coordinates (t, r) are the Rindler-type coordinates [9] associated with an accelerated observer in a flat spacetime. In this case the asymptotic correspondence between the two frames can not be established in the Schwarzschild coordinate system.

4 Concluding remarks

We have analyzed an analog of the Hadamard ansatz for the specification of local states associated with a linear quantum field conformally coupled to a two-dimensional gravitational background. For studying the constraints emerging from this ansatz we have employed a conformal invariant classical scalar field. The conformal invariance permits us to introduce a conformal frame in which the gravitational coupling of local states is established. Such a frame is constructed by relating the trace of the quantum stress-tensor to a cosmological constant. We have shown that the gravitational coupling of the local states leads to a two-dimensional gravitational theory.

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